Arithmetic differential invariants of dynamical systems

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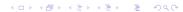
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 (X, σ) correspondence



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Define $\mathcal{O}(X)^{\sigma} = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

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If σ has a Zariski dense orbit then $\mathcal{O}(X)^{\sigma} = \mathbb{C}$

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Search for new geometries where we have invariants

Pass from the polynomial functions of algebraic geometry to more general functions called δ -functions

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Hope: more functions \Rightarrow more invariants

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In physics: no invariant functions on space-time under the symmetries of space-time but one has invariant Lagrangians; the same will happen in δ -geometry

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Want to apply δ -geometry to arithmetic geometry; but there are no derivations on \mathbb{Z} ; what to do?

The Fermat quotient $\boldsymbol{\delta}$

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The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N, p) = 1]^{\hat{}}$

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Example $f : \mathbb{A}^1(R) = R \to R$, $f(x) = \sum_{n \ge 1} p^n x^n (\delta x)^{n^3} (\delta^2 x)^{n^n}$, δ -function of order 2

$\delta\text{-line}$ bundles

Define δ -line bundle on X as a locally free sheaf of \mathcal{O}^r -modules of rank 1

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For $\sigma_1, \sigma_2: \widetilde{X} \to X$ etale between smooth schemes over R

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Define δ -line bundle on X as a locally free sheaf of \mathcal{O}^r -modules of rank 1 Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i : a_i \in \mathbb{Z}\}$ **Define** $W_+ = \{\sum a_i \phi^i; a_i \in \mathbb{Z}_+\}$ For $w = \sum a_i \phi^i \in W$, $f \in \mathcal{O}^r(X)^{\times}$ set $f^w = \prod (\phi^i(f))^{a_i}$. For L line bundle on X defined by cocycle (f_{ij}) define δ -line bundle L^w by cocycle (f_{ii}^w) For $\sigma_1, \sigma_2 : \tilde{X} \to X$ etale between smooth schemes over *R* $R_{\delta}(X, L) = \bigoplus_{0 \neq m \in W_{\perp}} H^{0}(X, L^{w})$ $R_{\delta}(X,L)^{\sigma} = \{s \in R_{\delta}(X,L); \beta \sigma_1^* s = \sigma_2^* s\}$ graded ring of δ -invariants $R_{\delta}(X,L)_{(0)}^{\sigma} = \{f/g; f,g \in R_{\delta}(X,L)^{h}, p \not| g, deg(f) = deg(g)\}, \delta$ -DVR of δ -invariants (δ acts on this)

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Basic Example $L = K^{-1}$, anticanonical bundle

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3) (hyperbolic case) The action of a Hecke correspondence on a modular (or Shimura) curve.

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Morally

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In all these cases the categorical quotient X/σ is a "rational variety in $\delta\text{-geometry}"$

Explanations

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post-critically finite with (orbifold) Euler characteristic zero is essentially equivalent to f multiplicative ($f(x) = x^N$), Chebyshev, or Lattèes (i.e. induced from an endomorphism of an elliptic curve)

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Converse result

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Theorem

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Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be defined over a number field and assume for p >> 0 the correspondence (X, σ) obtained from the graph of f satisfies $R_{\delta}(X, K^{-1})^{\sigma}_{(0)} \neq R$. Then f is post critically finite with (orb) Euler characteristic zero.

1) Construct δ -invariants



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Flat case: study dynamical systems with invariant tensor differential forms mod p (B-, IRMN 05)

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Theorem

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E/R elliptic curve. There exists a δ -function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

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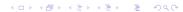
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$$\psi^2(nP) = n^2\psi(P), \psi^2(-P) = \psi^2(P)$$



$\delta\text{-modular}$ forms

 $X \subset X_1(N)$, N > 4, X affine, disjoint from cusps and supersingular locus

$\delta\text{-modular}$ forms

 $X \subset X_1(N)$, N > 4, X affine, disjoint from cusps and supersingular locus L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

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$\delta\text{-modular}$ forms

 $X \subset X_1(N), N > 4, X$ affine, disjoint from cusps and supersingular locus *L* line bundle on $X_1(N)$ whose *m*-power has sections modular forms of weight *m* $V^* = Spec(\bigoplus_{m \in \mathbb{Z}} L_X^m)$ physical line bundle on *X* minos zero section

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The generators

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Theorem

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Theorem

1. There exists $f^1 \in M^1(-1-\phi)$ with δ -Fourier expansion $\frac{1}{p}\log\left(1+p\frac{q'}{q^p}\right)$

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Rings of invariant functions for (X, σ) identified with rings of functions on X/σ

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Rings of invariant functions for (X, σ) identified with rings of functions on X/σ

A different approach to quotients: groupoid strategy. Comes in 2 flavors: 1) Grothendieck's descent and 2) Connes' NC-geometry. Rings of functions on X/σ replaced by 1) descent data or 2) convolution rings

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Grothendieck descent not strong enough to deal with correspondences that have dense orbits but Connes' NC-geometry strong enough to deal with some cases

Comparison between δ -geometry and *NC*-geometry

	δ -geometry	NC- geometry
spherical	$\frac{\mathbb{P}^1(R)}{SL_2(\mathbb{Z}_p)}$	$\frac{\mathbb{P}^1(\mathbb{R})}{\mathit{SL}_2(\mathbb{Z})} = NC\text{-modular}$ curve
flat	$\frac{E(R)}{\langle \gamma_i \rangle}, \frac{E(R)}{[n]}$	$rac{S^1}{\langle e^{2\pi i heta} angle} =$ NC-elliptic curve
hyperbolic	$\Gamma ackslash \mathbb{H} = Sh_{\Gamma} o rac{Sh_{\Gamma}}{ ext{Hecke}}$	$lim Sh_\Gamma = Sh^0 \subset Sh \subset Sh^{(nc)}$

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The 2 geometries are very different but they apply to similar situations

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Theorem (Poonen+B-, Compositio 2009)



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 $\Phi: X = X_1(N) \rightarrow A$ modular parametrization, A elliptic curve

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Theorem (Poonen+B-, Compositio 2009)

- $\Phi: X = X_1(N) \rightarrow A$ modular parametrization, A elliptic curve
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- $Q \in X(R)$ an ordinary point.

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Theorem (Poonen+B-, Compositio 2009)

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Similar results for Heegner points (C replaced by CL)

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Assume $\Gamma = A(R)_{tors}$ and C replaced by CL



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Consider $f^{\sharp} = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

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Assume $\Gamma = A(R)_{tors}$ and C replaced by CLConsider $f^{\sharp} = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2 $f^{\flat} : X(R) \subset X_1(N)(R) \to R$, $f^1|f^{\flat}$, $f^{\flat}(CL) = 0$, order 1

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Finitely many solutions (by Krasner's theorem) plus bound